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On the existence of multiple solutions of the boundary value problem for nonlinear second order differential equations

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We consider the second order ordinary differential equation

$$u'' + a(x)f(u) = 0, \quad 0 < x < 1 \tag{1.1}$$

with the boundary condition

$$u(0) = u(1) = 0. \tag{1.2}$$

In equation (1.1) we assume that a satisfies

$$a \in C^1[0,1], \quad a(x) > 0 \quad \text{for } 0 \le x \le 1, \tag{1.3}$$

and that f satisfies the following conditions (H1)–(H3):

(H1) $f \in C(\mathbf{R})$, $f(s) > 0$ for $s > 0$, $f(-s) = -f(s)$ for $s > 0$, and f is locally Lipschitz continuous on $(0, \infty)$;

(H2) There exist limits f_0 and f_∞ such that $0 \le f_0, f_\infty \le \infty$,

$$f_0 = \lim_{s \to +0} \frac{f(s)}{s} \quad \text{and} \quad f_\infty = \lim_{s \to \infty} \frac{f(s)}{s};$$

(H3) In the case where $f_0 = \infty$ in (H2), $f(s)$ is nondecreasing and $f(s)/s$ is nonincreasing on $(0, s_0]$ for some $s_0 > 0$.

From (H1) we see that $f(0) = 0$. The case where $f(s) = |s|^{p-1}s$ with $p > 0$ is a typical case satisfying (H1)–(H3). Thus, $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. While, if $0 < f_0 < \infty$ and $0 < f_\infty < \infty$, then f is asymptotically linear at 0 and ∞, respectively.

In this paper we investigate the existence of multiple solutions of the problem (1.1) and (1.2) in terms of the behavior of the ratio $f(s)/s$ near $s = 0$ and near $s = \infty$. This kind of problem has been studied by many authors with various methods and techniques. We refer for instance to the papers [1–9, 11–15, 17] and the references cited therein. The purpose of this paper is to improve the condition concerning the behavior of the ratio $f(s)/s$ in the several known results. Hence our results help us to treat the known results from a unified point of view, and to develop the previous arguments.

Let λ_k be the k-th eigenvalue of

$$\text{(1.4)} \qquad \begin{cases} \varphi'' + \lambda a(x)\varphi = 0, \quad 0 < x < 1, \\ \qquad \varphi(0) = \varphi(1) = 0, \end{cases}$$

and let φ_k be an eigenfunction corresponding to λ_k. It is known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and that φ_k has exactly $k-1$ zeros in $(0,1)$. (See, e.g., [16, Chap. IV, Sec. 27].) For convenience, we put $\lambda_0 = 0$.

First we consider the case where the range of $f(s)/s$ contains no eigenvalue of the problem (1.4).

Theorem 1. *Assume that there exists an integer $k \in \mathbf{N} = \{1, 2, \ldots\}$ such that*

$$\text{(1.5)} \qquad \lambda_{k-1} < \frac{f(s)}{s} < \lambda_k \quad \textit{for } s \in (0, \infty).$$

Then the problem (1.1) *and* (1.2) *has no solution $u \in C^2[0,1]$.*

Next we consider the case where the range of $f(s)/s$ contains at least one eigenvalue of the problem (1.4). Note that if u is a solution of (1.1), so is $-u$, because of $f(-s) = -f(s)$.

Theorem 2. *Assume that either $f_0 < \lambda_k < f_\infty$ or $f_\infty < \lambda_k < f_0$ for some $k \in \mathbf{N}$. Then the problem* (1.1) *and* (1.2) *has a pair of solutions u_k and $-u_k$ which have exactly $k-1$ zeros in $(0,1)$.*

Theorem 3. *Assume that either the following* (i) *or* (ii) *holds for some $k \in \mathbf{N}$:*

$$\text{(i)} \quad f_0 < \lambda_k < \lambda_{k+1} < f_\infty; \qquad \text{(ii)} \quad f_\infty < \lambda_k < \lambda_{k+1} < f_0.$$

Then the problem (1.1) *and* (1.2) *has pairs of solutions $\pm u_k$ and $\pm u_{k+1}$ such that u_k and u_{k+1} have exactly $k-1$ and k zeros in $(0,1)$, respectively, and satisfy $0 < u_k'(0) < u_{k+1}'(0)$ if* (i) *holds, and $u_k'(0) > u_{k+1}'(0) > 0$ if* (ii) *holds.*

Let us consider the cases where either f is superlinear or sublinear. As a consequence of Theorem 3 we obtain the following:

Corollary 1. *Assume that either the following* (i) *or* (ii) *holds:*

$$\text{(i)} \quad f_0 = 0, \; f_\infty = \infty; \qquad \text{(ii)} \quad f_0 = \infty, \; f_\infty = 0.$$

Then there exist pairs of solutions $\pm u_k$ $(k = 1, 2, \dots)$ *of the problem* (1.1) *and* (1.2) *such that* u_k *has exactly* $k-1$ *zeros in* $(0,1)$ *for each* $k \in \mathbf{N}$, *and that*

$$0 < u_1'(0) < u_2'(0) < \cdots < u_k'(0) < u_{k+1}'(0) < \cdots$$

if (i) *holds, and*

$$u_1'(0) > u_2'(0) > \cdots > u_k'(0) > u_{k+1}'(0) > \cdots > 0$$

if (ii) *holds.*

Remark. For the superlinear and sublinear cases, the existence of positive solutions of (1.1) and (1.2) has been obtained by Erbe and Wang [7] by using fixed point techniques.

The existence of an infinite sequence of solutions of (1.1) and (1.2) has been studied by Hartman [11] and Hooker [12] for the superlinear case, and Capietto and Dambrosio [3] for the sublinear case. In [11] and [12], Corollary 1 has been given under a weaker condition on a and f. For the superlinear and sublinear cases, we refer to [15].

Let us consider the nonlinear eigenvalue problem of the form

$$\begin{cases} u'' + \lambda a(x) f(u) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.6}$$

where $\lambda > 0$ is a real parameter. We assume in (1.6) that a satisfies (1.3) and f satisfies (H1)–(H3). By virtue of Theorem 2 we easily obtain the following result.

Corollary 2. *Assume that either the following* (i) *or* (ii) *holds*:

$$\text{(i)} \quad f_0 = 0, \ f_\infty = 1; \qquad \text{(ii)} \quad f_0 = 1, \ f_\infty = 0.$$

Assume, in addition, that $\lambda_k < \lambda < \lambda_{k+1}$ *for some* $k \in \mathbf{N}$, *where* λ_k *is the* k-*th eigenvalue of the problem* (1.4). *Then the problem* (1.6) *possesses* k *pairs of solutions* $\pm u_j$ $(j = 1, 2, \dots, k)$ *such that* u_j *has exactly* $j-1$ *zeros in* $(0,1)$.

Remark. Kolodoner [13] has shown that the problem

$$u'' + \lambda \frac{u}{\sqrt{x^2+u^2}} = 0, \quad u(0) = u'(1) = 0,$$

possesses k pairs of solutions provided $\lambda_k < \lambda \le \lambda_{k+1}$, where λ_k is the k-th eigenvalue of the linearized problem

$$\varphi'' + \lambda \frac{\varphi}{x} = 0, \quad \varphi(0) = \varphi'(1) = 0.$$

Dinca and Sanchez [6] have established a generalization of Kolodner's results.

By a change of variable, it can be shown that the existence of solutions of the problem (1.1) and (1.2) is equivalent to the existence of radial solutions of the following Dirichlet problem for semilinear elliptic equations in annular domains:

$$\Delta u + a(|x|)f(u) = 0 \quad \text{in } \Omega, \tag{1.7}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

where $\Omega = \{x \in \mathbf{R}^N : R_1 < |x| < R_2\}$, $R_1 > 0$ and $N \geq 2$. We assume in (1.7) that $a \in C^1[R_1, R_2]$, $a(r) > 0$ for $R_1 \leq r \leq R_2$, and that f satisfies conditions (H1)–(H3).

Let μ_k be the k-th eigenvalue of

$$\begin{cases} (r^{N-1}\phi')' + \mu r^{N-1}a(r)\phi = 0, & R_1 \leq r \leq R_2, \\ \phi(R_1) = \phi(R_2) = 0. \end{cases} \tag{1.9}$$

It is known (see, e.g., [16, Chap. IV, Sec. 27]) that

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_k < \mu_{k+1} < \cdots, \quad \lim_{k\to\infty} \mu_k = \infty.$$

From Theorems 1 and 2 and Corollary 1, we obtain the following result which will be proved in Section 3.

Corollary 3. (i) *Assume that there exists an integer $k \in \mathbf{N}$ such that*

$$\mu_{k-1} < \frac{f(s)}{s} < \mu_k \quad \text{for } s \in (0, \infty).$$

Then the problem (1.7) *and* (1.8) *has no radial solution $u(r)$ in $C^2[R_1, R_2]$, where $r = |x|$.*

(ii) *Assume that either $f_0 < \mu_k < f_\infty$ or $f_\infty < \mu_k < f_0$ for some $k \in \mathbf{N}$. Then there exists a radial solution $u_k(r)$ of the problem* (1.7) *and* (1.8) *which has exactly $k-1$ zeros in (R_1, R_2). In particular, if either* (i) *or* (ii) *in Corollary* 1 *holds, then there exist radial solutions $u_k(r)$ $(k = 1, 2, \ldots)$ of* (1.7) *and* (1.8) *such that $u_k(r)$ has exactly $k-1$ zeros in (R_1, R_2) for each $k \in \mathbf{N}$.*

Remark. The existence of radial positive solutions of (1.7) and (1.8) has been studied by many authors. For example we refer to [1, 2, 4, 5, 8, 9, 14] for the superlinear case, and to [17] for the sublinear case.

The existence of solutions with prescribed numbers of zeros is discussed by Coffman and Marcus [4] for the superlinear case.

Recently, Ercole and Zumpano [8] have established the existence of radial positive solutions with no assumptions on the behavior of the nonlinearity f either at zero or at infinity. They have used the fixed point theorem.

Theorem 1 follows immediately from the Sturm–Picone theorem. The proofs of Theorems 2 and 3 depends on the shooting method combined with the Sturm's comparison theorem. Namely we consider the solution $u(x;\mu)$ of (1.1) satisfying the initial condition

$$u(0) = 0 \quad \text{and} \quad u'(0) = \mu,$$

and observe the number of zeros of $u(x;\mu)$ in $(0,1]$ when $\mu \to 0$ and $\mu \to \infty$, by using the Sturm's comparison theorem. Here $\mu \in \mathbf{R}$ is a parameter.

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